

Snake Lemma

For $n \in \mathbb{N}$, let A_n denote the set of strings of length n comprising the letters A–Z. Let $A = \bigcup_{n \in \mathbb{N}} A_n$, and let $W \subseteq A$ denote the set of English words.

For convenience, define $f_0 = f_{26}$. All diagrams commute.

$$W_n = \lim \left(\begin{array}{c} W \\ \downarrow \iota \\ A_n \xrightarrow{\iota} A \end{array} \right)$$

$$\begin{array}{ccc} A & \xrightarrow{f_m} & A \\ c_n \downarrow & & \downarrow c_n \\ A & \xrightarrow{f_{(m+n) \bmod 26}} & A \end{array}$$

$$A \xrightarrow[c_{13}]{} A \xrightarrow[c_{13}]{} A \quad \iota$$

$$A \xrightarrow[f_5]{\iota} A \xrightarrow[f_4]{u_1} A$$

$$A \xrightarrow[f_9]{\iota} A \xrightarrow[f_8]{u_{50}} A$$

$$\begin{array}{ccccc} A & \xrightarrow{u_{13}} & A & \xrightarrow{u_{31}} & A \\ & \searrow f_9 & & & \nearrow f_3 \\ & A & \xrightarrow{u_{25}} & A & \end{array}$$

$$\begin{array}{ccccc}
A & \xrightarrow{o} & A & \xrightarrow{f_n} & A \\
& \searrow f_m & & & \nearrow o \\
& A & \xrightarrow{f_n} & A &
\end{array}$$

$$\begin{array}{ccc}
A_1 & \xrightarrow{\iota} & A \\
& \searrow \iota & \downarrow o \\
& A &
\end{array}$$

$$\begin{array}{ccccc}
A & \xrightarrow{o} & A & \xrightarrow{u_{21}} & A \\
& \searrow u_{18} & & & \nearrow o \\
& A & \xrightarrow{u_{43}} & A &
\end{array}$$

$$\begin{array}{ccc}
A_k \times A_k & \xrightarrow{f_n \times f_n} & A_{k+1} \times A_{k+1} \\
\downarrow \omega & & \downarrow \omega \\
A & \xrightarrow{f_n} & A
\end{array}$$

If $m \neq n$, then

$$\begin{array}{ccc}
A_k \times A_k & \xrightarrow{f_m \times f_n} & A_{k+1} \times A_{k+1} \\
& \searrow \omega & \downarrow \omega \\
& A &
\end{array}$$

$$\begin{array}{ccc}
A_k \times A_k & \xrightarrow{u_{15} \times u_{34}} & A_{k+2} \times A_{k+2} \\
\downarrow \omega & & \downarrow \omega \\
A & \xrightarrow{f_{11}} & A
\end{array}$$

Products satisfy a universal property:

$$\begin{array}{ccccc}
& & Y & & \\
& \swarrow \varphi_1 & \downarrow \exists! & \searrow \varphi_2 & \\
X_1 & \xleftarrow[\pi_1]{} & X_1 \times X_2 & \xrightarrow[\pi_2]{} & X_2
\end{array}$$

$$\begin{array}{ccccc}
& & A & & \\
& \swarrow \iota & \downarrow \Delta & \searrow \iota & \\
A & \xleftarrow{\pi_1} & A \times A & \xrightarrow{\pi_2} & A
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \times A \\
& \searrow \iota & \downarrow \omega \\
& & A
\end{array}$$

$$\begin{array}{ccc}
A_{2n+1} & \xrightarrow{o} & A \\
r \downarrow & & \uparrow r \\
A & \xrightarrow{o} & A
\end{array}$$

$$P = \lim \left(A \xrightarrow[\iota]{r} A \right)$$

$$P_n = \lim \left(\begin{array}{c} P \\ \downarrow \iota \\ A_n \xrightarrow{\iota} A \end{array} \right)$$

$$P_2 = \lim \left(\begin{array}{ccc} & & A_2 \\ & & \downarrow \iota \\ A_1 & \xrightarrow{\Delta} & A_1 \times A_1 & \xrightarrow{\kappa} & A \end{array} \right)$$

$$\begin{array}{ccccccc}
A_{2n} & & & & & & \\
\Delta \downarrow & & & & \iota & & \\
A_{2n} \times A_{2n} & \xrightarrow[\iota \times r]{} & A_{2n} \times A_{2n} & \xrightarrow[o \times o]{} & A_n \times A_n & \xrightarrow[\iota \times r]{} & A_n \times A_n \xrightarrow[\rho]{} A_{2n}
\end{array}$$

$$\begin{array}{ccccccc}
P_{2n} & & & & & & \\
\Delta \downarrow & & & & \iota & & \\
P_{2n} \times P_{2n} & \xrightarrow{o \times o} & A_n \times A_n & \xrightarrow[\iota \times r]{} & A_n \times A_n & \xrightarrow[\rho]{} & A
\end{array}$$

$$\begin{array}{ccccc}
A_n \times A_n & \xrightarrow{u_{32} \times u_{17}} & A_{n+2} \times A_{n+2} & \xrightarrow{\rho} & A \\
& \searrow \rho & & \nearrow u_{24} & \\
& A & \xrightarrow[u_9]{} & A &
\end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{f_m} & A \\ u_n \downarrow & & \downarrow u_n \\ A & \xrightarrow{i_{2,m}} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{u_{22}} & A \\ & \searrow u_{18} & \downarrow t_2 \\ & & A \end{array}$$

If $k \geq \max\{m, n\}$, then

$$\begin{array}{ccccc} & & A_k & \xrightarrow{t_n} & A_k \\ & \nearrow t_m & & & \searrow t_m \\ A_k & & & & A_k \\ & \searrow t_n & & & \nearrow t_n \\ & & A_k & \xrightarrow{t_m} & A_k \end{array}$$

$$\begin{array}{ccc} A_m \times A & \xrightarrow{\iota \times f_n} & A_m \times A \\ i_{m,n} \times \iota \downarrow & & \downarrow \kappa \\ A \times A & \xrightarrow[\kappa]{} & A \end{array}$$

$$\begin{array}{ccccc} A \times A & \xrightarrow{u_{47} \times \iota} & A \times A & \xrightarrow{r \times \iota} & A \times A \\ r \times u_{32} \downarrow & & & & \downarrow \kappa \\ A \times A & \xrightarrow[\kappa]{} & A & & A \end{array}$$

$$L_{n,m} = \lim \left(A \xrightarrow[i_{n,m}]{} A \right)$$

$$\begin{array}{ccc} L_{n,m} & \xrightarrow{d_n} & A \\ & \searrow i_{n,m} & \downarrow \iota \\ & & A \end{array}$$

$$P_{2n} = \lim \left(\begin{array}{ccc} & & P_{2n} \\ & \downarrow & \downarrow \iota \\ P_{2n-1} & \xrightarrow{d_n} & A \end{array} \right)$$

$$\begin{array}{ccc} A & \xrightarrow{s_5} & A \\ c_{22} \downarrow & & \uparrow c_4 \\ A & \xrightarrow{u_{46}} & A \xrightarrow{i_{2,8}} A \end{array}$$

$$\begin{array}{ccccc} A_n \times A_n & \xrightarrow{e_{90} \times e_4} & A_{n+2} \times A_{n+2} & \xrightarrow{\rho} & A \\ \rho \searrow & & & & \nearrow e_{65} \\ & A & \xrightarrow{e_2} & A & \end{array}$$

$$\begin{array}{ccccc} P \times A & \xrightarrow{e_{68} \times \iota} & A \times A & \xrightarrow{r \times \iota} & A \times A \\ \iota \times s_2 \downarrow & & & & \downarrow \kappa \\ P \times A & \xrightarrow{\kappa} & A & & \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{p_{55}} & A \\ u_{21} \downarrow & & \uparrow e_{35} \\ A & \xrightarrow{u_{48}} & A \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{p_{48}} & A & & \\ f_{16} \downarrow & & & & \uparrow t_6 \\ A & \xrightarrow{f_{14}} & A & \xrightarrow{e_{57}} & A \xrightarrow{s_1} A \end{array}$$

Theorem. Let

$$X = \lim \left(\begin{array}{c} W_3 \times W_2 \times W_4 \\ \downarrow \iota \times \kappa \\ W_3 \times A \\ \downarrow \iota \times c_{14} \\ W_3 \times A \\ \downarrow s_6 \times f_{20} \\ A \times A \\ \downarrow \iota \times e_{102} \\ A \times A \\ \downarrow \iota \times d_3 \\ A \times A \\ \downarrow r \times r \\ A \times A \\ \downarrow \iota \times p_{98} \\ A \times A \\ \downarrow e_{11} \times t_{12} \\ W \times W \xrightarrow{\iota \times \iota} A \times A \end{array} \right),$$

and let $\varphi : X \rightarrow W \times W$ be the canonical map coming from the limit structure. Then the answer to this puzzle is $\varphi(X)$.