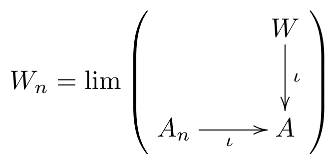
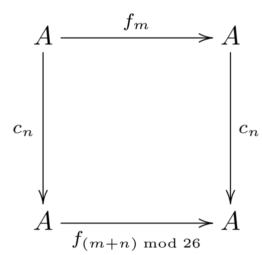
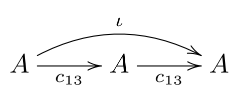
Snake Lemma

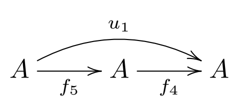
For $n \in \mathbb{N}$, let A_n denote the set of strings of length n comprising the letters A–Z. Let $A = \bigcup_{n \in \mathbb{N}} A_n$, and let $W \subseteq A$ denote the set of English words.

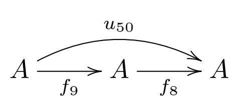
For convenience, define $f_0 = f_{26}$. All diagrams commute.

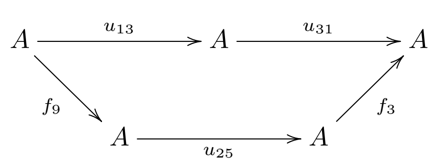


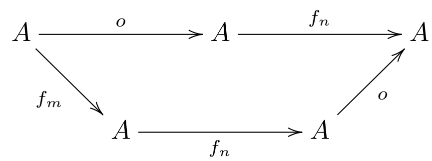


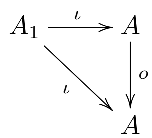


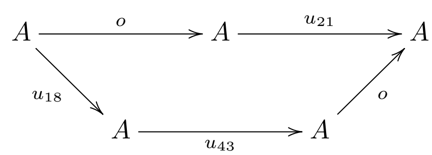


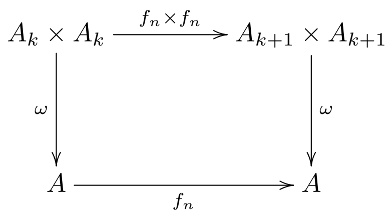




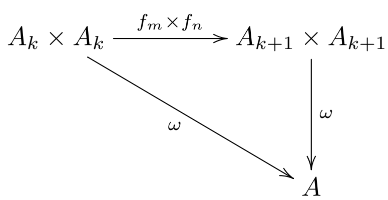


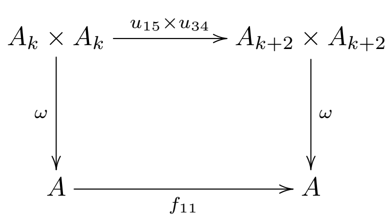




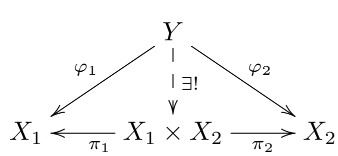


If $m \neq n$, then





Products satisfy a universal property:



$$
\begin{array}{ccccc}
 & & A & & \\
 & \swarrow^{\iota} & \downarrow{\scriptstyle \Delta} & \searrow^{\iota} & \\
A & \xleftarrow[\pi_1]{} & A \times A & \xrightarrow[\pi_2]{} & A
\end{array}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \times A \\
 & \searrow_{\iota} & \downarrow{\scriptstyle \omega} \\
 & & A
\end{array}
$$

$$
\begin{array}{ccc}
A_{2n+1} & \xrightarrow{o} & A \\
{\scriptstyle r}\downarrow & & \uparrow{\scriptstyle r} \\
A & \xrightarrow[o]{} & A
\end{array}
$$

$$
P = \lim \left(A \underset{\iota}{\overset{r}{\rightrightarrows}} A \right)
$$

$$
P_n = \lim \left(\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \iota} \\ A_n & \xrightarrow[\iota]{} & A \end{array} \right)
$$

$$
P_2 = \lim \left(\begin{array}{ccccc} & & & & A_2 \\ & & & & \downarrow{\scriptstyle \iota} \\ A_1 & \xrightarrow{\Delta} & A_1 \times A_1 & \xrightarrow{\kappa} & A \end{array} \right)
$$

$$
\begin{array}{ccccccccccc}
A_{2n} & & & & & \xrightarrow{\iota} & & & & & A_{2n} \\
{\scriptstyle \Delta}\downarrow & & & & & & & & & & \\
A_{2n} \times A_{2n} & \xrightarrow[\iota \times r]{} & A_{2n} \times A_{2n} & \xrightarrow[o \times o]{} & A_n \times A_n & \xrightarrow[\iota \times r]{} & A_n \times A_n & \xrightarrow[\rho]{} & A_{2n} & &
\end{array}
$$

$$
\begin{array}{ccccccc}
P_{2n} & & & \xrightarrow{\iota} & & & A \\
{\scriptstyle \Delta}\downarrow & & & & & & \\
P_{2n} \times P_{2n} & \xrightarrow[o \times o]{} & A_n \times A_n & \xrightarrow[\iota \times r]{} & A_n \times A_n & \xrightarrow[\rho]{} & A
\end{array}
$$

$$
\begin{array}{ccccc}
A_n \times A_n & \xrightarrow{u_{32} \times u_{17}} & A_{n+2} \times A_{n+2} & \xrightarrow{\rho} & A \\
 & \searrow_{\rho} & & \nearrow_{u_{24}} & \\
 & A & \xrightarrow[u_9]{} & A &
\end{array}
$$

$$\begin{array}{ccc} A & \xrightarrow{f_m} & A \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle u_n} \\ A & \xrightarrow[i_{2,m}]{} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{u_{22}} & A \\ & {\scriptstyle u_{18}}\searrow & \downarrow{\scriptstyle t_2} \\ & & A \end{array}$$

If $k \geq \max\{m, n\}$, then

$$\begin{array}{ccccc} & A_k & \xrightarrow{t_n} & A_k & \\ {\scriptstyle t_m}\nearrow & & & & \searrow{\scriptstyle t_m} \\ A_k & & & & A_k \\ {\scriptstyle t_n}\searrow & & & & \nearrow{\scriptstyle t_n} \\ & A_k & \xrightarrow[t_m]{} & A_k & \end{array}$$

$$\begin{array}{ccc} A_m \times A & \xrightarrow{\iota \times f_n} & A_m \times A \\ {\scriptstyle i_{m,n} \times \iota}\downarrow & & \downarrow{\scriptstyle \kappa} \\ A \times A & \xrightarrow[\kappa]{} & A \end{array}$$

$$\begin{array}{ccccc} A \times A & \xrightarrow{u_{47} \times \iota} & A \times A & \xrightarrow{r \times \iota} & A \times A \\ {\scriptstyle r \times u_{32}}\downarrow & & & & \downarrow{\scriptstyle \kappa} \\ A \times A & & \xrightarrow[\kappa]{\qquad\qquad} & & A \end{array}$$

$$L_{n,m} = \lim \left(A \underset{i_{n,m}}{\overset{i_{n-1,m}}{\rightrightarrows}} A \right)$$

$$\begin{array}{ccc} L_{n,m} & \xrightarrow{d_n} & A \\ & {\scriptstyle i_{n,m}}\searrow & \downarrow{\scriptstyle \iota} \\ & & A \end{array}$$

$$P_{2n} = \lim \left(\begin{array}{ccc} & & P_{2n} \\ & & \downarrow{\scriptstyle \iota} \\ P_{2n-1} & \xrightarrow[d_n]{} & A \end{array} \right)$$

$$\begin{array}{ccccc} A & & \xrightarrow{s_5} & & A \\ {\scriptstyle c_{22}}\downarrow & & & & \uparrow{\scriptstyle c_4} \\ A & \xrightarrow[u_{46}]{} & A & \xrightarrow[i_{2,8}]{} & A \end{array}$$

$$\begin{array}{ccccc} A_n \times A_n & \xrightarrow{e_{90} \times e_4} & A_{n+2} \times A_{n+2} & \xrightarrow{\rho} & A \\ & {\scriptstyle \rho}\searrow & & \nearrow{\scriptstyle e_{65}} & \\ & A & \xrightarrow[e_2]{} & A & \end{array}$$

$$\begin{array}{ccccc} P \times A & \xrightarrow{e_{68} \times \iota} & A \times A & \xrightarrow{r \times \iota} & A \times A \\ {\scriptstyle \iota \times s_2}\downarrow & & & & \downarrow{\scriptstyle \kappa} \\ P \times A & & \xrightarrow[\kappa]{} & & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{p_{55}} & A \\ {\scriptstyle u_{21}}\downarrow & & \uparrow{\scriptstyle e_{35}} \\ A & \xrightarrow[u_{48}]{} & A \end{array}$$

$$\begin{array}{ccccccc} A & & & \xrightarrow{p_{48}} & & & A \\ {\scriptstyle f_{16}}\downarrow & & & & & & \uparrow{\scriptstyle t_6} \\ A & \xrightarrow[f_{14}]{} & A & \xrightarrow[e_{57}]{} & A & \xrightarrow[s_1]{} & A \end{array}$$

Theorem. *Let*

$$X = \lim \left(\begin{array}{ccc} & & W_3 \times W_2 \times W_4 \\ & & \downarrow{\scriptstyle \iota \times \kappa} \\ & & W_3 \times A \\ & & \downarrow{\scriptstyle \iota \times c_{14}} \\ & & W_3 \times A \\ & & \downarrow{\scriptstyle s_6 \times f_{20}} \\ & & A \times A \\ & & \downarrow{\scriptstyle \iota \times e_{102}} \\ & & A \times A \\ & & \downarrow{\scriptstyle \iota \times d_3} \\ & & A \times A \\ & & \downarrow{\scriptstyle r \times r} \\ & & A \times A \\ & & \downarrow{\scriptstyle \iota \times p_{98}} \\ & & A \times A \\ & & \downarrow{\scriptstyle e_{11} \times t_{12}} \\ W \times W & \xrightarrow[\iota \times \iota]{} & A \times A \end{array} \right),$$

and let $\varphi : X \to W \times W$ be the canonical map coming from the limit structure. Then the answer to this puzzle is $\varphi(X)$.